Technology as a partner in geometry classrooms

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Abstract

In this paper we present technology-based activities in geometry that promote pre-service teachers' mathematical reasoning. These future teachers explored carefully structured activities that allowed engaging technological skills as well as developing and evaluating geometric conjectures. The technology was used as a partner in both of the examples presented here and the students explored each activity through paper folding, technology application and constructing proofs as part of reflexive pedagogy in action. As teacher educators, we recognized technology as an important part of developing pre-service teachers' professionally situated knowledge.

1. Introduction

In this article we propose technology-based activities in geometry that are suitable for promoting pre-service teachers' mathematical reasoning at middle school and high school levels. The two carefully structured activities, one organized at teacher education program at a Canadian university and another in an American, provided opportunities for dealing with students' misconceptions, and for developing and evaluating geometric conjectures while engaging with newly acquired technological skills. In both examples, the pre-service teachers explored each problem through manual construction, technology application, and formal proof as part of reflexive pedagogy in action. As teacher educators, we recognized technology as an important factor in developing pre-service teachers' professionally situated knowledge—the technology was used as a partner to increase students' power to investigate, explore, and communicate.

In their 2010 paper, *Mathematical Knowledge and Practices Resulting from Access to Digital Technologies*, Olive et al. [1] suggested that the new digital technologies bring to new levels didactic practices that were traditional for mathematics classrooms. The authors proposed that the usual didactic triangle, that has the teacher, mathematical knowledge, and student as vertices, should also include technology. Consequently, the model changes from a triangle to a tetrahedron, which defines the 3-D space "in which new mathematical knowledge and practices can emerge" ([1], page 133). In that way, technology adds yet another dimension into mathematics teaching and learning.

Similarly, Mishra and Koehler [2], expanded Shulman's [3] original construct of pedagogical content knowledge (PCK) as one type of teacher knowledge. Mishra and Koehler present their framework as a Venn diagram that joins three overlapping types of knowledge such as (a) knowledge of technology (TK), (b) knowledge of pedagogy (PK), and (c) knowledge of content (CK). As a result, the authors proposed a technological pedagogical content knowledge (TPCK) as an additional construct to be considered. As the overlap between the three knowledges, TPCK presents the thoughtful approach to integration of technology into the classroom in order to support content learning.

Finally, technology can take different roles, depending on a pedagogical context and type of technology. Researchers discussed four levels of use of technology: 1) technology as a master; 2) technology as a servant; 3) technology as a partner; and, 4) technology as an extension of self [4], [5].

In this hierarchical model, technology is used as a master when either mathematical or technological skills are lacking. If the former is the case, one may "believe" or take for granted any output produced by software, lacking capability to engage his/her mathematical thinking to evaluate the product. If the latter is the case, one may be adept mathematically, but use technology inadequately or superficially. On the other hand, technology is a servant when used non-creatively, more as "an obedient but dumb assistant" ([4], page 312). Technology becomes a partner when it is used hand-in-hand with mathematical reasoning. If this is the case, the user is critical of results produced by technology and relies on his/her mathematical knowledge to make a judgement. The most advanced application of technology is as extension of self. In this case, technology becomes part of mathematical repertoire in the same fashion as graphs and tables, and is used seamlessly. While Goos, et al. [4] define the model as increasing in sophistication, they do not claim it is developmental. Depending on a context, the person whose previous use of technology was on a higher level may use it as a servant, for example [5].

In the following examples, technology was used as a partner for its motivational purposes to increase students' power to investigate, explore, and communicate. Technology introduced the element of surprise and ambiguity in a problem solving situation and as such motivated students to employ their logical proof strategies [6]. These two teaching situations exemplify Hadas, Hershkowitz, and Schwarz's [7] notion that technology may make formal proofs necessary, rather than make them redundant; may change the position of students from the recipients of formal proofs, into the seekers of ways to overcome contradictions and uncertainties. Our motivation, as teacher educators, was to present future teachers with interesting geometric problems suitable for various levels of schooling, to demonstrate use of different representations (e.g., visual, symbolic, physical), and to explore the role that technology may have in building their professionally situated knowledge.

2. A famous Dudeney's (1907) hinged triangle-to-square dissection problem

The following exercise was done with the Intermediate/Senior (I/S)[1] group of pre-service teachers. These future teachers were educated to work with diverse student populations, different curricula, and classroom settings. For these reasons, the instruction in the mathematics methods course incorporated manipulatives, computer software, interactive boards, and more traditional approaches. The exercise was employed as a quick and powerful introduction into the dynamic mathematics software (e.g., GeoGebra, The Geometer's Sketchpad). The activity was related to a famous Dudeney's [7] hinged triangle-to-square dissection problem (see Figure 1). It built on the theorem that any polygon can be transformed into another polygon of the same area, by cutting and rearranging the pieces [8]. By adding

[1] The I/S pre-service teachers in Ontario, Canada, prepare for teaching Grades 7-12.

a restriction that all pieces of a dissection must be connected by hinges, the problem becomes more difficult to the extent that "it is an open problem whether any two polygons of equal area have a swing-hinged dissection in a finite number of pieces" ([8], page 245). However, the hinged equilateral triangle-to-square dissection exists and still sparks interest after more than a century since Dudeney [8] created its physical model.

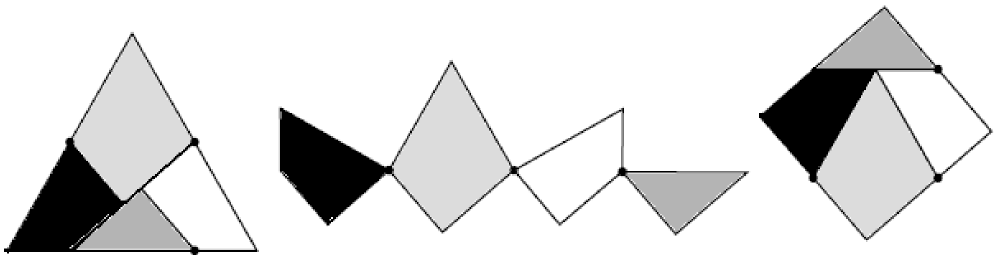


Figure 1. *Dudeney's ([8]) hinged triangle-to-square dissection.*

The activity presented to pre-service teachers consisted of four stages: Stage 1—that is, folding, cutting and reassembling a rectangular piece of paper; Stage 2—performing similar moves in the dynamic mathematics software, which produced an unexpected result (the reconstructed quadrilateral is not a square); Stage 3—where after doing a proof, the students became convinced that their original intuition was false; and Stage 4—an extension, where students were encouraged to find another approach to solving this problem.

2.1 Stage 1— Creating a manipulative[2]

Instructions: To square a triangle, you need a rectangular sheet of paper, a pencil, several rulers, and a pair of scissors. This construction takes 15 steps (see Figure 2, with snapshots taken during the activity).

How to make an equilateral triangle with a paper sheet: Take a standard paper sheet and fold it lengthwise in two equal parts as shown in step 2. Unfold the paper sheet, fold back a corner of the paper sheet on the fold mark and draw a line along the corner side (see step 4). Repeat this operation with the opposite corner of the paper sheet. Cut the sheet along the two drawn lines (step 7) to get an equilateral triangle.

How to transform your equilateral triangle into a square: Fold back the three corners of the triangle to make a rectangle as in step 9. Then, unfold the triangle; the fold marks define a rectangle HIGF. Trace the diagonal HG of the rectangle HIGF (step 11). With the help of rulers trace two perpendiculars to the diagonal HG, one passing through point F, the other one through point I, as shown in step 12. The triangle is now divided in four pieces. Cut them out and reassemble them as shown in step 15.

[2] Adapted from http://www.archimedes-lab.org/workshoptriangledissect.html

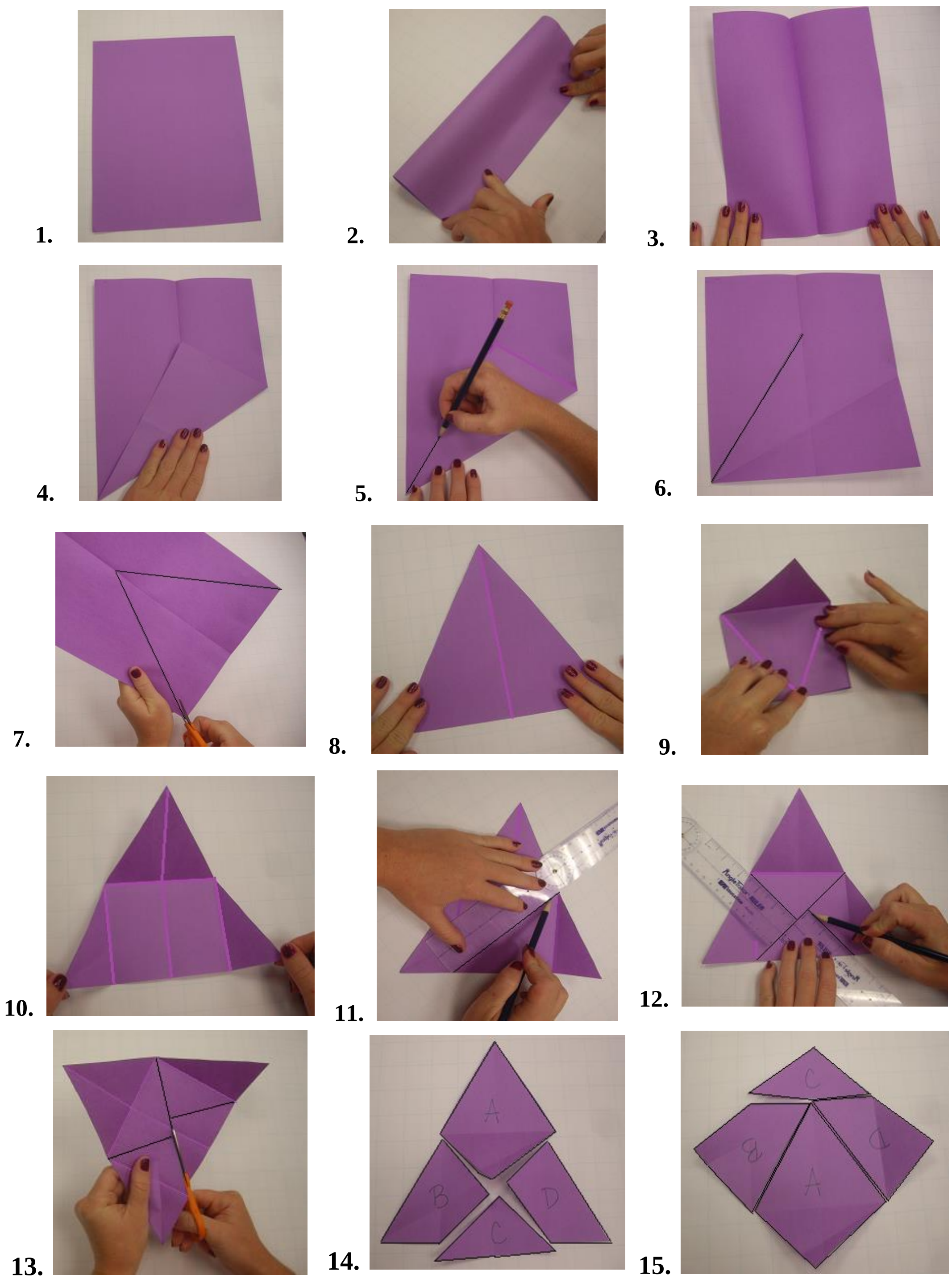


Figure 2: *instructions for paper folding*[3].

[3]Instructions are available on http://www.archimedes-lab.org/workshoptriangledissect.html

In conclusion of the paper-folding activity, students posed the conjecture that the final product was a square. As a follow up, students did the same exercise in GeoGebra by performing a series of geometric constructions and transformations. Here, we are showing the construction starting from the equilateral triangle in step 8 of Figure 2.

2.2 Stage 2 — Replication of paper folding activity in GeoGebra

In GeoGebra, there is a tool for constructing a regular polygon. After constructing a segment between two arbitrary points A and B, the students used the regular pentagon tool. This tool allowed for construction of any regular polygon with a given number of sides. By selecting the points A and B on the base, and entering 3 as a number of sides, an equilateral triangle was constructed. The folding of a paper was further simulated through, (a) construction of a height CD, (b) construction of a midpoint E on CD, and (c) construction of a parallel line through E to the base and its intersections with sides of the triangle (to create a midsegment of the triangle).

The rest of the paper folding simulation was done through a series of similar steps that included construction of perpendicular lines in order to get a rectangle HIGF, and then construction of perpendicular lines to its diagonal HG. Figure 3 shows simulation of steps 8-12 of the paper folding exercise.

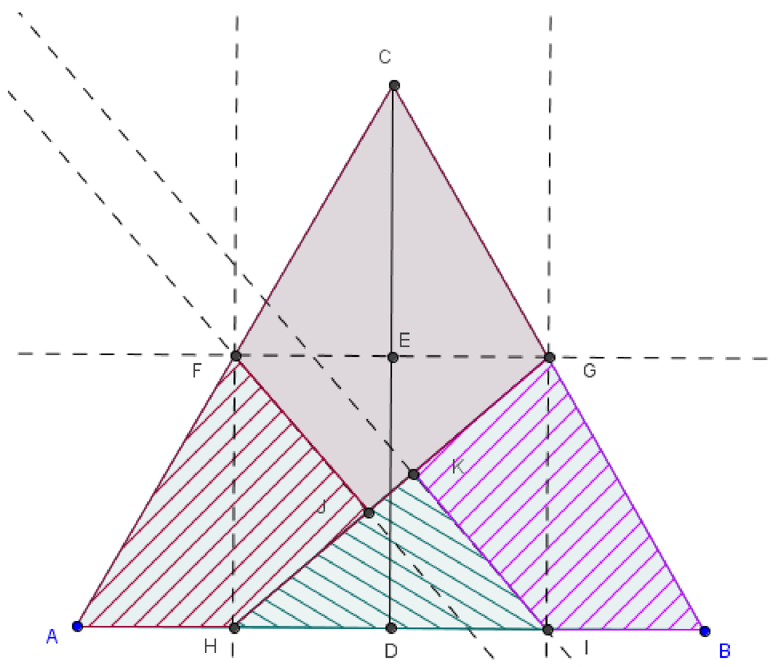


Figure 3: s*imulation of paper folding paper steps 8-12 on Figure 2.*

In the final step of the GeoGebra activity, students were invited to measure the lengths of the sides of the created quadrilateral OJLQ. They realized that the object may not be a square, but a rectangle. At this point, the students were puzzled that their conjecture was not correct. They questioned whether the difference between the sides of the quadrilateral was due to the rounding error or the nature of construction producing a rectangle?

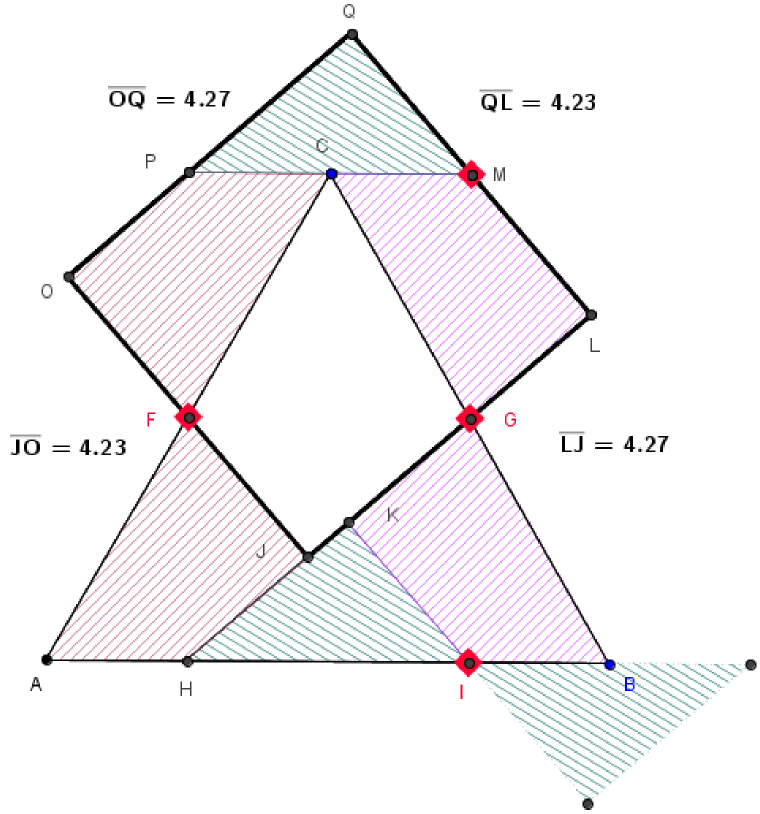


Figure 4: *simulation of reconstructing a quadrilateral OJLQ from the triangle ABC (hinges I, G, F, and M show in red).*

2.3 Stage 3 — A brief proof that the paper folding activity produced a rectangle

The question whether the difference between the sides of the quadrilateral was due to the rounding error or the nature of construction producing a rectangle was answered through the deductive process of proof.

Given: $\overline{AB} = \overline{BC} = \overline{CA} = a$, Prove: $\overline{JO} < \overline{JL}$

$\overline{HI} = \overline{FG} = \frac{a}{2}$ (FG is a midsegment of ΔABC)

$\overline{IG} = \frac{\overline{DC}}{2}$ (distance of the midsegment from the base is half of the height)

$\overline{DC} = \frac{a\sqrt{3}}{2}$ (as height in equilateral triangle)

$\overline{HG} = \frac{a\sqrt{7}}{4}$ (from the Pythagorean Theorem, as the hypotenuse of a right triangle with $\overline{IG} = \frac{a\sqrt{3}}{4}$ and $\overline{HI} = \frac{a}{2}$ as its legs)

AHJF $\cong$ BGKI (from the construction)

$\overline{\mathrm{HJ}} = \overline{\mathrm{KG}} = \overline{\mathrm{GL}} \Rightarrow \overline{\mathrm{JL}} = \overline{\mathrm{HG}} = \frac{a\sqrt{7}}{4}$
$\overline{\mathrm{JF}} = \overline{\mathrm{FO}}$ (from the construction)
$\overline{JF} = \frac{a\sqrt{21}}{14}$ (from ΔHFG)
$\overline{JO} = 2\overline{JF} = \frac{a\sqrt{21}}{7}$
$\overline{JO} < \overline{JL}$ (since $\frac{a\sqrt{21}}{7} < \frac{a\sqrt{7}}{4}$) Q.E.D.

From the construction it followed that the re-assembled quadrilateral had all four inner angles equal to 90^o and had opposite sides equal. Therefore, the quadrilateral OJLQ was a rectangle.

The technology in this example acted as a partner because it allowed students to examine their original conjectures and to question their assumptions. Based on the Stage 1 activities, students concluded that the paper folding produced a square. After finishing the technology portion of the activity during Stage 2, they were unsure if the outcome was a square. Therefore, students had to engage in deductive proof procedure in order to identify whether the created object was a square or a rectangle. As it is showed above, students proved that the resulting quadrilateral was indeed a rectangle.

2.4 Stage 4 — Finding a new way of constructing a square from the equilateral triangle

As a follow-up, students can adhere to a different set of instructions, one which was more aligned with what was given in the literature as a correct triangle-to-square dissection.

2.4.1 Instructions:

1. Find the area of an equilateral triangle ΔABC. If its side is a, then its height is $h = \frac{a\sqrt{3}}{2}$. Using the formula for the area of triangle, one gets that $A = \frac{a^2\sqrt{3}}{4}$. Since the square will have the same area as the original triangle, it is easy to find that its side must be $x = \frac{a\sqrt[4]{3}}{2}$ (i.e., $x = \sqrt{A}$).
2. Mark the midpoints of two sides of the triangle. Label the midpoints D and E (see Figure 5).
3. Set a compass radius at $x = \frac{a\sqrt[4]{3}}{2}$. Put the point of the compass on one of the midpoints, say E. Swing an arc and mark a point one radius away on the bottom (unmarked) edge of the triangle. Label this point F. Connect points E and F with a straight line.
4. Set the compass to a radius which is the length from D to E. Put the point of the compass on F, and swing an arc that cuts the bottom edge of the triangle. Label this point H.
5. Now, construct a segment that goes through point D and is perpendicular to line FE, ending on line FE. Label the point G where the line intersects FE.
6. Construct a second line segment that goes through point H and is perpendicular to line FE. Label the point I where the line intersects line FE. The triangle should look like in Figure 5.

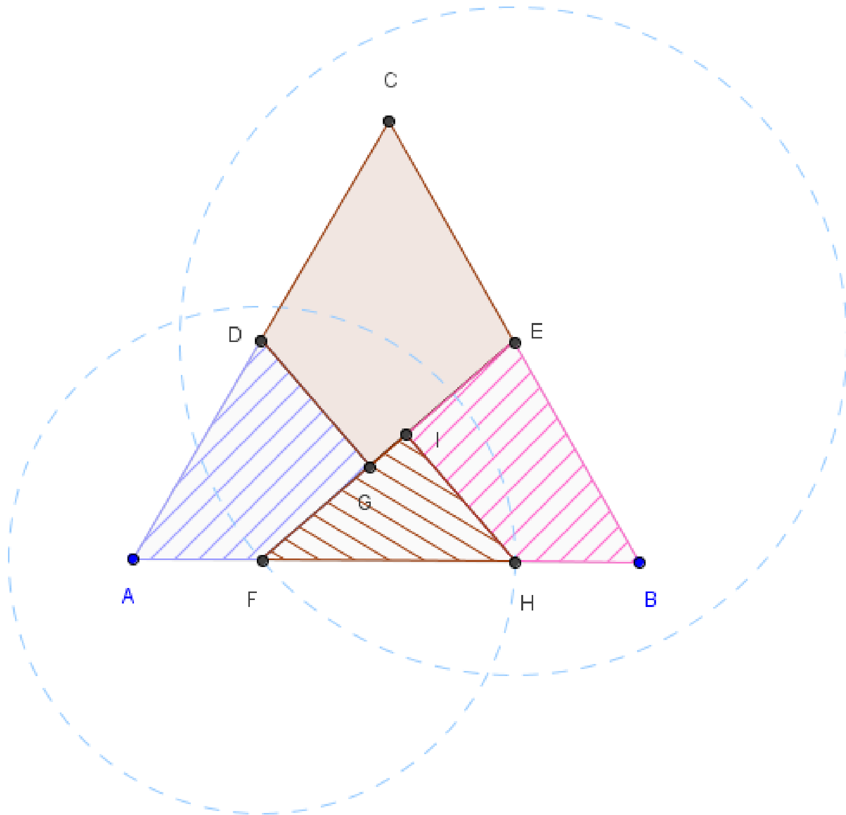


Figure 5: *the result of the equilateral triangle dissection*[4].

The instructor had the opportunity to make a point that the second construction was focused on the properties of square. Therefore it produced the square of equal area to the triangle, as a result of re-composition.

2.4.2 Recomposing a square using technology

These instructions were carried out in GeoGebra. Unfolding of the parts of the triangle was performed through rotations for 180^{O} clockwise of AFGD around D, and counter clockwise for 180^{O} of IFH, around H. These steps were followed by rotating a quadrilateral EIHB and a dotted triangle counter clockwise for 180^{O} around E (see Figure 6).

[4] See http://www.math.nmsu.edu/~breakingaway/Lessons/T2S/Triangle2Square.htm

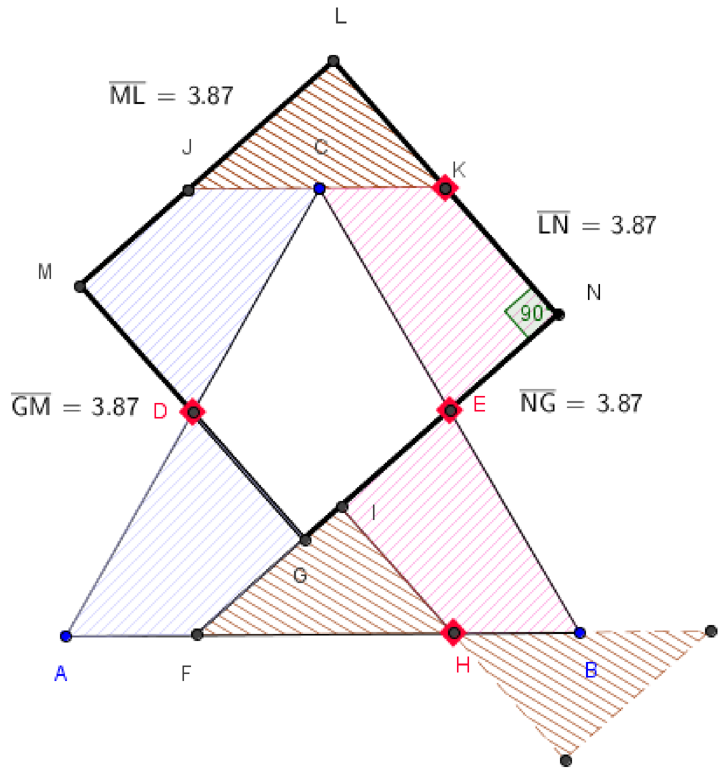


Figure 6: *completed unfolding of pieces of the triangle into the square (hinges show in red).*

The measurements of the rectangle produced in Stage 2 of the activity were very close to the measurements of the square produced in Stage 4; $\frac{a\sqrt{7}}{4} > \frac{a\sqrt[4]{3}}{2}$, where the fraction on the left is the length of the diagonal in the paper folding activity, while the fraction on the right is the length of the side of the square used to construct the distance EF in the Figure 6. These two numbers differ at the second decimal (.661>.658), which explains why the rectangle in Stage 2 was perceived as a square by the students.

The paper folding activity allowed students to create and check their conjectures after having faced a mathematical situation. What started as a fun but easy exercise for the future high school mathematics teachers, turned out to be a complex enough activity that only some could complete without the instructor's help. The students at the lower level of their geometric development could efficiently replicate the paper folding stage in the software and seamlessly be able to measure the angles and lengths, without going through calculations. However, most of the students were inclined to think that the exercise was completed even before doing the measurements in 2.2. Indeed, such exercise would then serve mostly to show them how to use some features of the software. For students that would mean using technology as a master or a servant. But the activity did not stop there; questions like, "How do we know that this is a square?" inspired most of the students to look for evidence that the quadrilateral is a square. Even the weaker students were puzzled after finding that the product of folding activity may be a rectangle. Interesting discussions began—"Shall we trust the instruction sheet, or the software? Is the difference in length between the sides of the quadrilateral real or accidental? How do we know? What would mathematicians do in such situation?" The partnership with technology [4] started when the students begun to question their original conjectures and to examine the alternative, not only technology-based, ways of solving the problem. Students became engaged in a continuous re-examination of their assumptions and ideas, especially about the role that technology has in this activity [4], [5]. In the end, to find the mathematically correct solution, the students had to go through the process of the formal proof. The role of the instructor was to guide students in all four stages of the activity and motivate them to engage with its reflexive nature.

The fact that these students were pre-service teachers means that they were particularly motivated to either avoid or resolve ambiguities in teaching situations. Most of the pre-service teachers are not sure in their PK, CK, or TK [2], which makes activities such as described in this paper especially valuable; they allow the instructor to shift the emphasis between the pedagogy, content, and technology, thus making the case that teachers need to feel comfortable with all three. That does not mean that teachers need to know all three, but that they need to develop a skill of tapping into one, two, or all three knowledges, depending on a situation (see Figure 7).

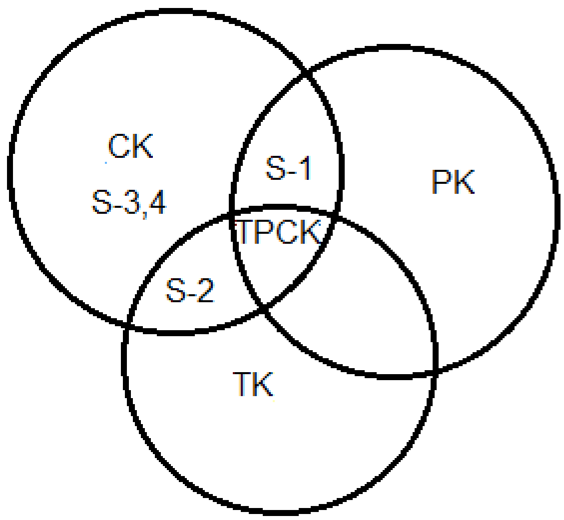


Figure 7: *moving within and between the knowledges during the stages of the folding activity.*

The Stage 1 (S-1), creating a manipulative, had a pedagogical aspect as a hands-on fun activity suitable for even the small children, however, it also tapped into the geometry of polygons and their transformations; the Stage 2 (S-2), replication of paper folding activity in GeoGebra, served to introduce dynamic geometry software to pre-service teachers, but also allowed for reviewing geometry concepts, such as points, lines, areas, rotations and translations; the Stages 3-4 (S-3,4), a proof that the paper folding activity produced a rectangle and constructing a square from the equilateral triangle, were inherently mathematical. In that way, only in one activity, the pre-service teachers shifted within and between the types of knowledge very relevant for teaching mathematics with technology. If such experiences were repeated throughout the course, they could influence the teaching practices of these pre-service teachers thus producing new mathematical knowledge and practices [1].

3. Triangle-to-parallelogram dissection problem

The second exercise was done with a group of the pre-service middle and high school teachers (Grades 6-12) at the small university located in the South-East region of the US. These future teachers were taught to work with wide range of classroom settings, diverse student populations, and were exposed to the variety of the instructional methods. They were presented with a paper-folding activity and were asked to compare areas of two shapes [9]. Students worked in small groups and developed the following two solutions for a given problem (see Figure 8).

Problem: Given a parallelogram and a triangle that has the same height and has doubled base, prove by dissecting the triangle that both shapes have the same areas.

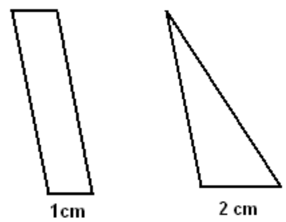


Figure 8: *instructions for triangle-to-parallelogram dissection problem.*

This activity consisted of three stages: Stage 1—folding, cutting, and re-assembling a triangular piece of paper into a parallelogram (see Figures 9 and 10); Stage 2—performing similar moves in the dynamic mathematics software which exposed an error in one of the solutions (see Figures 11 and 12); Stage 3—didactic reasoning and proof to justify solutions presented in Stages 1 and 2 (see Figures 13 and 14).

3.1 Stage 1: Dissecting a triangle.

3.1.1 Students' Solution 1 of Dissecting a Triangle

One group of students presented a strategy that had a mathematical error. They had dissected a triangle by sketching a median to the base of the triangle. Then they moved one of the created triangles into a position that produced what they perceived as a parallelogram (see Figure 9). Students claimed that area of the newly created geometric object is equal to the area of the parallelogram because it is constructed of two triangles with the same base and height as the parallelogram's. However, the dissection and the following decomposition did not produce a parallelogram, which was not obvious from the folding and cutting activity. The students attributed discrepancy to a human error.

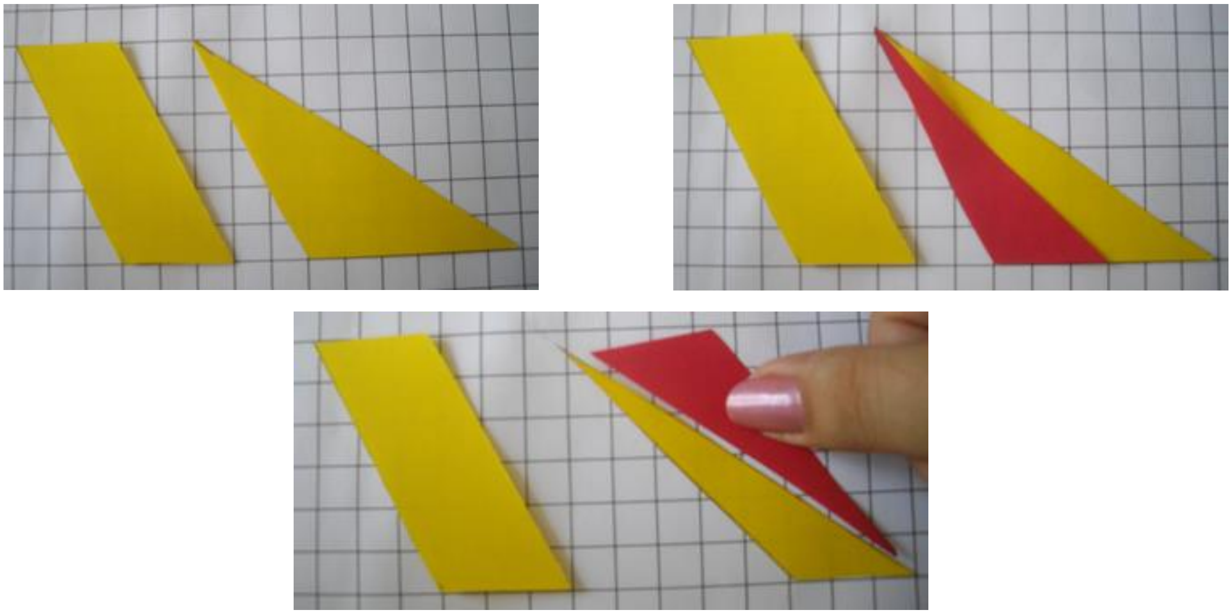


Figure 9: *the manipulative created by Group 1.*

3.1.2 Students' Solution 2 of Dissecting a Triangle

Another group of students produced a mathematically correct strategy. They decided to preserve the side and angle in the triangle that were congruent to the side and the angle of the parallelogram. Then they found a midpoint on the base and drew the line through the midpoint parallel to the preserved side of the triangle. This produced a triangle that was then moved to complete the parallelogram congruent to the original one.

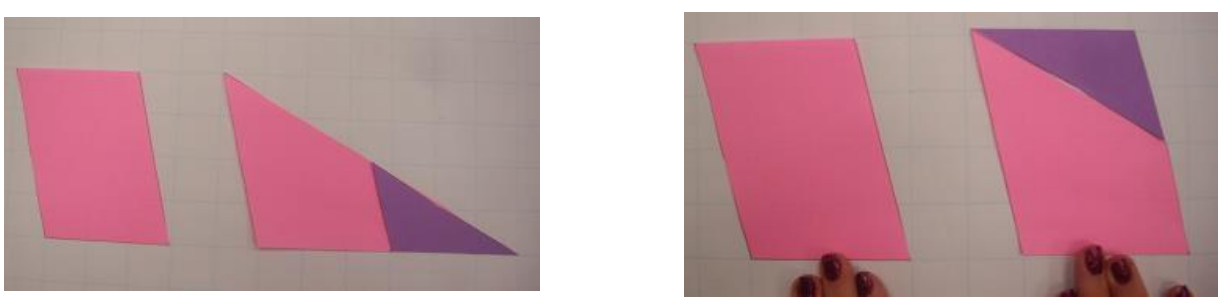


Figure 10: *the manipulative created by Group 2.*

In conclusion of the paper folding activity, both groups of students posed a conjecture that the final product of dissection and decomposition of the triangle produced a congruent parallelogram.

Next, the students were asked to do the same exercise in GeoGebra by performing geometric constructions and transformations.

3.2 Stage 2 — Replication of triangle dissecting activity in GeoGebra

3.2.1 Technology replication of Students' Solution 1:

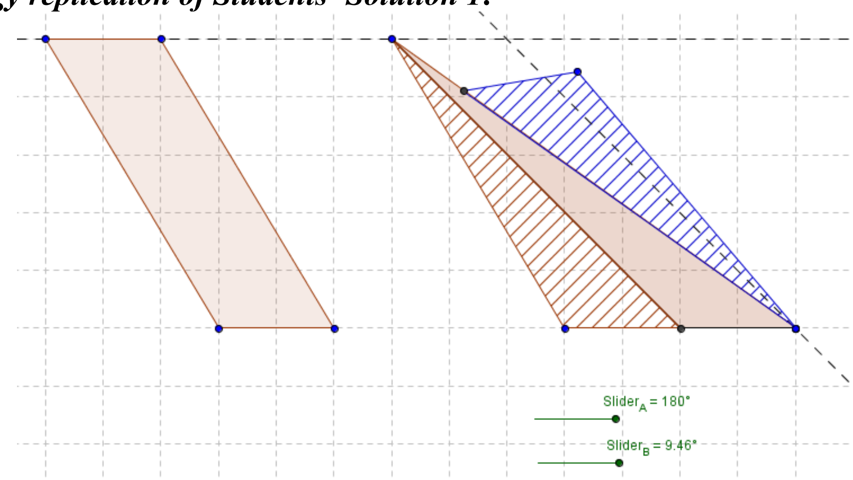


Figure 11: *diagram accompanying the Group 1 solution.*

When creating a GeoGebra sketch, students can easily measure areas of different polygons. In the sketch provided in Figure 13, the students exposed a misconception that the triangle BAD is congruent to triangle EAC. Students noticed that these two triangles have the same areas; however, they missed the fact that those triangles are not congruent. Due to the wrong assumption this group of pre-service teachers concluded that when moving triangle ABD to the location of the triangle CEA, they created a parallelogram CDAE that is congruent to the original parallelogram. The group was then asked to use the technology to either measure the sides and angles of the triangles and/or construct a GeoGebra transformation that would move triangle BAD into the new location. Students realized there was an error in their mathematical reasoning as is shown in Figure 11. After interacting with the technology construction they were convinced this method could not result in a parallelogram (see the attached dynamic GeoGebra file)[5].

3.2.2 Technology replication of Students' Solution 2:

[5] Link to GeoGebra file titled Final Student Solution 1.

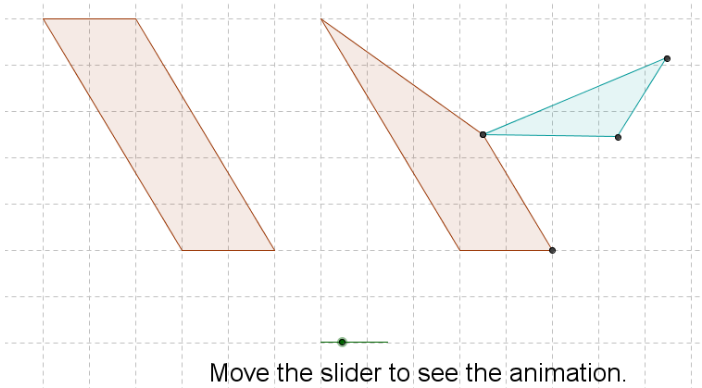


Figure 12: *diagram accompanying the Group 2 solution.*

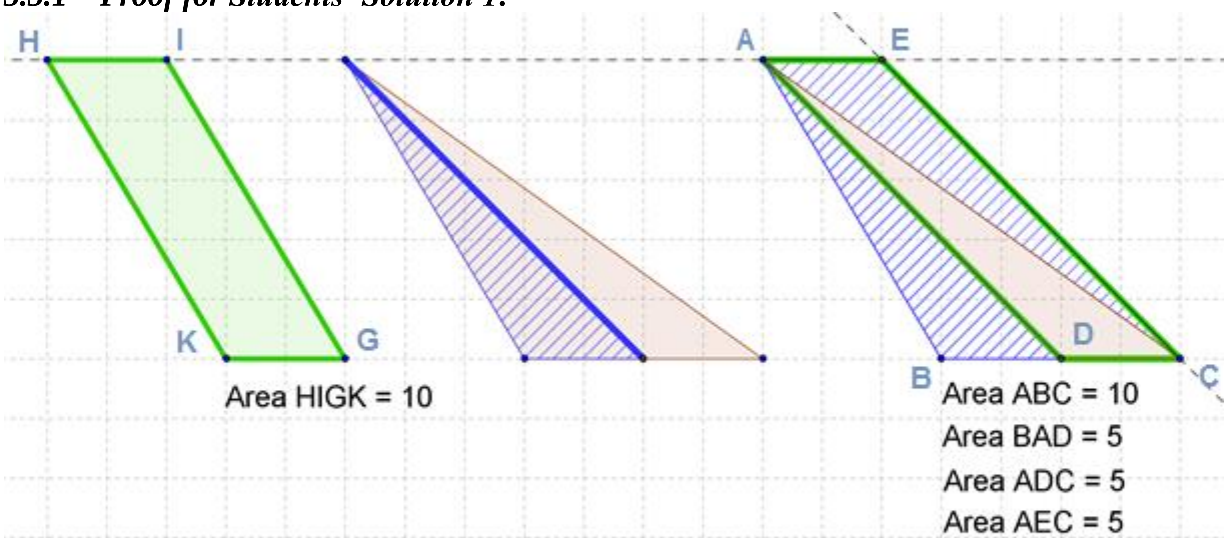
From the GeoGebra construction the second group of pre-service teachers was able to justify that the triangles CDE and ADF were congruent (see Figure 14) and therefore the rotation of the triangle CDE around point D by 180^o would produce a parallelogram congruent to the original one (see the attached dynamic GeoGebra file)[6].

Both groups of students used technology as a partner when examining an error in the first case and justifying the conjecture in the second case.

3.3 Stage 3 — Using proof for Group 1 and Group 2 solutions

Students in Groups 1 and 2 were asked to justify their reasoning using formal proof. Group 1 of pre-service teachers was able to see the mathematical error in their reasoning after completing this portion of the activity (see below).

3.3.1 Proof for Students' Solution 1:



[6] Link to GeoGebra file titled Final Student Solution 2.

Figure 13: *justification of the proof of Group 1.*

Find D as a midpoint of $\overline{\text{BC}}$.
$\overline{\text{BD}} = \overline{\text{DC}}$
ΔABD and ΔADC have the same height $\overline{\text{AG}}$ and congruent bases. Therefore these triangles have the same area.

However, ΔABD is not congruent to ΔADC as it does not meet any congruence conditions. When dissecting and relocating ΔABD as shown above it does not create a parallelogram. In this example, the triangle and parallelogram have the same areas but dissection does not produce congruent shapes. The instructor could extend this problem by asking students to think of special cases in which their solution would work. This would allow future teachers to take advantage of the reflexive nature of this problem as an opportunity for exploring more mathematics.

3.3.2 Proof for Students' Solution 2:

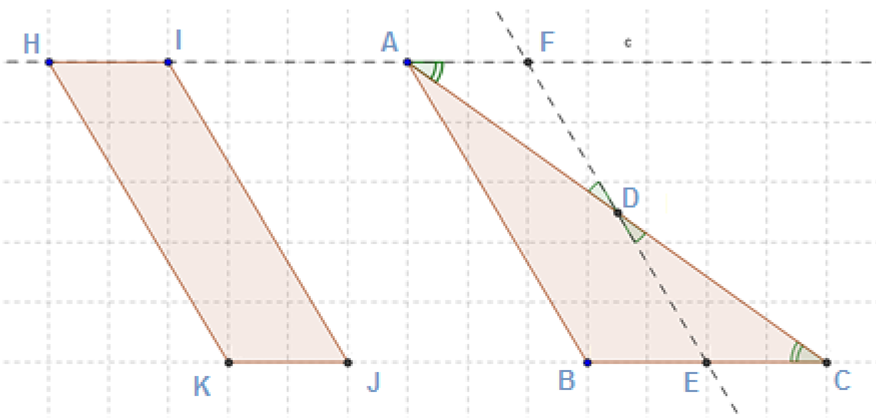


Figure 14: *justification of the proof of Group 2.*

Find E so that it is a midpoint of $\overline{BC}$, $\overline{BE} = \overline{EC}$
Sketch line b through point E so that b || $\overline{AB}$
$\Delta DCE \approx \Delta ACB \Rightarrow \overline{CD} = \overline{DA}$
Sketch line c through point A so that c || $\overline{BC}$; $\angle ADF = \angle CDE$ as vertically opposite angles
$\angle FAD = \angle ECD$, since $\overline{AF}$ || $\overline{EC}$ (as alternate angles)
$\Delta ADF \cong \Delta CDE$ (angle-side-angle) This means that $\overline{AF} = \overline{CE}$, therefore $\overline{AF} = \overline{BE}$.
We can conclude that ABEF is a parallelogram, and is congruent to HKJI.

In this activity students were given an opportunity to create conjectures using paper folding activity. Then pre-service teachers were asked to test their conjectures using dynamic geometry software as a partner in this task. In both cases of Students' Group 1 and Students' Group 2 solutions, they first attempted to manually dissect a triangle and re-compose it into a new parallelogram that is congruent to the original parallelogram. Both groups then made a claim that since parallelograms are congruent their areas must be the same. In fact, in the second solution, dissection and the following transformation produced the shape that was congruent to the original parallelogram. However, in the first solution, the dissection and the following re-composition did not produce a congruent parallelogram. In this case, students noticed the incongruity in the model and contributed this discrepancy to the human error. Technology application allowed direct comparison of the shapes and their attributes. Students were able to see that incongruity was not a simple human error and recognised need for further mathematical investigation. Mathematical proofs were the way how both groups were able to see the difference in their approaches and the error in the first group's solution. However without partnering with technology students in the first group would not been able to recognise error in their thinking. Technology in this case allowed pre-service teachers to think critically and re-evaluate their original assumptions.

As technological tools advance, learners have an opportunity to discover and explore more mathematics. Ultimately, students have to reason mathematically. By performing formal proofs students were able to examine their conjectures in both examples presented in this article.

Finally, instructors used these activities to provide pre-service teachers with opportunities to increase their subject matter knowledge and to improve their pedagogical and technological knowledge. They accomplished this goal by modelling the type of teaching that focuses on development of conceptual knowledge of mathematics. In both activities described in the article instructors modelled questions that have to be asked to promote students' mathematical reasoning. These questions included: "How do we know if this shape is a parallelogram? What do we know about attributes of constructed shapes? Compare strategies presented by your peers…" Pre-service teachers also observed ways instructors used technology in a problem-rich environments. These experiences in a teacher training program can influence future teachers' PK, CK, and TK.

4. Discussion and conclusion

The activities described in this paper could be done with students at different levels of schooling. In the pre-service program they would contribute to the development of the pre-service teachers into reflexive pedagogues. Teachers need to be reflective practitioners. According to Finlay and Gough ([10], page ix), reflection is "thinking about" something after the event, while reflexivity is a more immediate, dynamic and continuing self-awareness. Each activity described here had an emphasis on developing self-awareness as a learner, user of technology, and future teacher, and therefore provided opportunities for improving professionally situated knowledge of pre-service teachers.

In addition, the examples given here demonstrated what Olive et al. [1] call a role of feedback in practice. The feedback pre-service teachers received from the software alerted them to inaccuracies in their solutions; the actions that followed were thus in reaction to what the computer had produced. In the first method for solving a Dudeney's [7] hinged triangle-to-square dissection problem, the difference between the sides of the right angled quadrilateral, perceived as a square, was so small that it could be attributed to several sources (e.g., rounding error in software) and ignored. Indeed, at this point the instructor had to step in and let the students raise their concerns. As a result, pre-service teachers looked for alternative solution paths and acknowledged the importance of implementing a formal proof strategy. This approach exemplified what Zbiek and Glass [11] term extensive and deep reasoning that happens when one is confronted with a conflicting result produced by technology. Technology thus became their partner in the investigation process [4], [5].

The use of technology as a partner described here allowed pre-service teachers to pose conjectures and examine validity of these conjectures. These examples also demonstrated the importance of the formal proof and its place in the pre-service teacher training. They showed how to avoid a trap described by Hanna [12] and Hoyles and Jones [13] of replacing a formal proof by explorations in software. Hanna [12] argued that exploring and proving should be considered as separate but complementary tools of mathematicians; "[e]xploration leads to discovery, while proof is confirmation" ([12], page 14). In our examples, exploration in software allowed pre-service teachers to discover a possible error in the dissection; the error which was confirmed by formal proofs.

It is important to highlight the value of "carefully designed tasks, professional teacher input, and opportunities for students to notice details, to conjecture, to make mistakes, to reflect, to interpret relationships among objects, and to offer tentative mathematical explanations" ([12], page 21). In both problems described here, the students were ready to stop their work multiple times during each lesson. In the Dudeney's [7] hinged triangle-to-square dissection problem, the pre-service teachers were ready to accept the result of a manual manipulation as a solution of the problem. The simulation of a manipulative creation in dynamic geometry software was also acceptable as a proof, since the rectangle really "looked" like a square. In the dissection of a triangle into parallelogram example, some pre-service teachers were ready also to accept the wrong solution after establishing that two triangles have the same area which was the area of half of the parallelogram. With careful and tactful, but persistent, nudge from the instructor, the pre-service teachers took another step, and then another, becoming increasingly excited with the learning that took place.

Finally, presenting aforementioned examples allowed pre-service teachers to examine own thinking about the mathematics and possible student errors in technology-based activities in the classroom. This was important when developing technological pedagogical content knowledge as well as the subject matter knowledge. Dudeney's [7] hinged triangle-to-square and triangle-to-parallelogram dissection problems provided rich opportunities for pre-service teachers to increase their reasoning skills, engage in reflexive pedagogy, and improve their technological pedagogical content knowledge (TPCK).

Dissection problems presented here also demonstrate how technology may help the instructor to create horizontal and vertical connections between curriculum units, see for example [14], where vertical connections are done to the prior knowledge and horizontal connections are made among multiple representations, such as symbolic, physical, and visual. While there are other forms and possibilities for using technology as a partner in learning and doing mathematics (see a review of intelligent partnership with technology in [15]), the instructors are in the dire need for examples that worked well and that could be readily implemented. Our hope is that both the new and seasoned teachers will use technology as a partner in their mathematics classrooms, and that their students will experience the beauty and importance of mathematical reasoning through geometry examples provided in this paper.

5. References

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Software Packages

GeoGebra http://www.geogebra.org/cms/

Supplemental Electronic Materials

Dudeney Folding Triangle
Final Student Solution 1
Final Student Solution 2